4.4.4 Transformation of Velocities (Addition of Velocities)

Suppose, relative to a frame S, a particle has a velocity

$$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k} \tag{4.51}$$

where $u_x = dx/dt$ etc. What we require is the velocity of this particle as measured in the frame of reference S' moving with a velocity v_x relative to S. If the particle has coordinate x at time t in S, then the particle will have coordinate x' at time t' in S' where

$$x = \gamma()x' + v_x t') \text{ and } t = \gamma(t' + v_x x'/c^2). \tag{4.52}$$

If the particle is displaced to a new position $x + dx$ at time $t + dt$ in S, then in S' it will be at the position $x' + dx'$ at time $t' + dt'$ where

$$x + dx = \gamma(x' + dx' + v_x(t' + dt'))$$

$$t + dt = \gamma\left(t' + dt' + v_x(x' + dx')/c^2\right)$$

and hence

$$dx = \gamma(dx' + v_x dt')$$

$$dt = \gamma(dt' + v_x dx'/c^2)$$

so that

$$u_x = \frac{dx}{dt} = \frac{dx' + v_x dt'}{dt' + v_x dx'/c^2}$$

$$= \frac{\frac{dx'}{dt'} + v_x}{1 + \frac{v_x}{c^2}\frac{dx'}{dt'}}$$

$$= \frac{u'_x + v_x}{1 + v_x u'_x/c^2} \tag{4.53}$$

where $u'_x = dx'/dt'$ is the X velocity of the particle in the S' frame of reference. Similarly, using $y = y'$ and $z = z'$ we find that

$$u_y = \frac{u'_y}{\gamma(1 + v_x u'_x/c^2)} \tag{4.54}$$

$$u_z = \frac{u'_z}{\gamma(1 + v_x u'_x/c^2)}. \tag{4.55}$$

The inverse transformation follows by replacing $v_x \to -v_x$ interchanging the primed and unprimed variables. The result is

$$\left.\begin{aligned} u'_x &= \frac{u_x - v_x}{1 - v_x u_x/c^2} \\ u'_y &= \frac{u_y}{\gamma(1 - v_x u_x/c^2)} \\ u'_z &= \frac{u_z}{\gamma(1 - v_x u_x/c^2)}. \end{aligned}\right\} \tag{4.56}$$

In particular, if $u_x = c$ and $u_y = u_z = 0$, we find that

$$u'_x = \frac{c - v_x}{1 - v_x/c} = c \tag{4.57}$$

i.e., if the particle has the speed c in S, it has the same speed c in S'. This is just a restatement of the fact that if a particle (or light) has a speed c in one frame of reference, then it has the same speed c in all frames of reference.

Now consider the case in which the particle is moving with a speed that is less that c, i.e. suppose $u_y = u_z = 0$ and $|u_x| < c$. We can rewrite Eq. (4.56) in the form

$$\begin{aligned} u'_x - c &= \frac{u_x - c}{1 - u_x v_x/c^2} - c \\ &= \frac{(c + v_x)(c - v_x)}{c(1 - v_x u_x/c^2)}. \end{aligned} \tag{4.58}$$

Now, if S' is moving relative to S with a speed less than c, i.e. $|v_x| < c$, then along with $|u_x| < c$ it is not difficult to show that the right hand side of Eq. (4.58) is always negative i.e.

$$u'_x - c < 0 \text{ if } |u_x| < c,\ |v_x| < c \tag{4.59}$$

from which follows $u'_x < c$.

Similarly, by writing

$$\begin{aligned} u'_x + c &= \frac{u_x - v_x}{1 - u_x v_x/c^2} + c \\ &= \frac{(c + u_x)(c - v_x)}{c(1 - v_x u_x/c^2)} \end{aligned} \tag{4.60}$$

we find that the right hand side of Eq. (4.60) is always positive provided $|u_x| < c$ and $|v_x| < c$ i.e.

$$u'_x + c > 0 \text{ if } |u_x| < c,\ |v_x| < c \tag{4.61}$$

from which follows $u'_x > -c$. Putting together Eq. (4.59) and Eq. (4.61) we find that

$$|u'_x| < c \text{ if } |u_x| < c \text{ and } |v_x| < c. \tag{4.62}$$

What this result is telling us is that if a particle has a speed less than c in one frame of reference, then its speed is always less than c in any other frame of reference, provided this other frame of reference is moving at a speed less than c. As an example, consider two objects A and B approaching each other, A at a velocity $u_x = 0.99c$ relative to a frame of reference S, and B stationary in a frame of reference S' which is moving with a velocity $v_x = -0.99c$ relative to S.